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Weak-anchoring effects on a Freedericksz transition in an annulus

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This paper investigates the effect of weak anchoring on static orientation patterns in a sample of nematic liquid crystal confined to a cylindrical annular geometry, in the presence of a magnetic field. The particular arrangement considered in detail is that in which a magnetic field is applied in the azimuthal direction to a sample of liquid crystal that is initially uniformly aligned parallel to the cylindrical axis and is weakly anchored at the inner cylinder, but strongly anchored at the outer cylinder. A static solution of the non-linear continuum equations is presented which suggests a possible experiment for evaluating the surface elastic coefficient k_{24}. In addition, a linear stability analysis based on the dynamic theory yields results for critical phenomena in agreement with those derived from the non-linear solution.

1. Introduction

In recent years there have been a number of studies concerning the orientation patterns of nematic liquid crystals confined in cylindrical geometries when there is weak anchoring at the boundaries. In particular, such studies have led to the first measurements of the surface elastic constant k_{24} by Crawford and co-workers (see, for example [1], and references therein). They used NMR techniques to ascertain the orientation patterns of nematics in sub-micrometer cylindrical cavities (namely, micropores in membranes), from which they could infer values of k_{24} (and of anchoring coefficients). Values of k_{24} have also been determined from optical measurements of orientation patterns in super-micrometer cavities (Polak *et al.* [2]), and from measurements of periodic patterns in hybrid aligned layers (Sparavigna *et al.* [3]). These studies indicate that k_{24} is of the same order as the bulk elastic constants. Further means of determining k_{24} have been proposed by Kralj and Žumer [4], based on detailed calculations of energy states of nematics in cylindrical cavities.

In this paper we describe a possible alternative arrangement for measuring k_{24}, based on the observation of Freedericksz transitions in a cylindrical annulus. Of pertinence to this study are the analyses of Leslie [5] and of Atkin and Barratt [6], which consider a nematic sample in an annulus, strongly anchored at the boundaries. Leslie investigates the case in which the initial orientation is everywhere azimuthal with a magnetic field applied radially outwards, while Atkin and Barratt examine the case when the initial orientation is uniformly aligned parallel to the common axis, and the magnetic field is applied in either the azimuthal or radial direction. Both papers demonstrate that as the field strength exceeds a critical value, a distorted configuration becomes available which is energetically favourable compared to the initial orientation pattern. Thus one anticipates that the distorted solution occurs in preference to the uniform alignment, and hence a Freedericksz transition results. The relationships obtained in the analyses between the critical field strength and various material parameters provide an experimentalist with possible methods for either determining information about the Frank elastic constants k_{11}, k_{22} and k_{33} or checking the continuum theory of Ericksen [7] and Leslie [8].

Rapini and Papoular [9] proposed a simple form for the surface free energy per unit area of a nematic, and examined the effect of weak anchoring on the classical Freedericksz transition in a sample confined by parallel flat plates. Jenkins and Barratt [10] employed a variational principle to obtain the condition for balance of couple at a nematic–liquid boundary by endowing the interface with a rather general surface free energy per unit area. Barratt [11] used these ideas in proposing a model for disclination lines in cylindrical samples of nematics; he found that k_{24} appears in the balance-of-couple condition. Palffy-Muhoray *et al.* [12] have recently presented results for saddle-splay and mechanical instabilities in nematics confined to a cylindrical annular geometry. They demonstrate that configurations in which the alignment is either everywhere azimuthal or everywhere radial become unstable when the outer radius is increased beyond some critical value. In the event of weak anchoring on one boundary and strong anchoring on the other, they observe

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that a critical relation exists which involves k_{24} and various anchoring coefficients, as well as the cylindrical radii. An obvious difficulty in using their results to determine k_{24} (by observing critical phenomena) is the practicality of increasing the outer radius continuously.

Here we consider a similar experimental arrangement to that of Palffy-Muhoray *et al.* [12], but involving the application of a magnetic field; this overcomes the above-mentioned difficulty of obtaining critical values in their arrangements. To this end, we investigate the problem in which the liquid crystal is confined between two coaxial circular cylinders, with the initial orientation uniformly parallel to the common axis of the cylinders, and a magnetic field is applied in the azimuthal direction. After outlining in § 2 the continuum theory required for the ensuing analysis, we present in § 3 a novel exact static solution of the non-linear equations when there is weak anchoring on the inner cylinder and strong anchoring on the outer cylinder. For ease in calculations we employ a particularly simple formulation for the interfacial energy to model the weak anchoring condition. It is found that no distortion of the initial configuration occurs until the applied field strength exceeds a critical value which is dependent upon the elastic constant k_{24} as well as other parameters. The constant k_{24} may be positive or negative, and the type of solution obtained in § 3 is found to be dependent on its sign. In § 4, we perform a linear stability analysis of the uniform static configuration with respect to time dependent perturbations in both the director and velocity fields. It is shown that in this case the principle of exchange of stabilities holds and one can recover the critical relation obtained in § 3. In addition we consider the effect of adopting a more general expression for the interfacial energy.

Commonly, threshold values for Freedericksz transitions have been derived via minimization of free energy (using Euler–Lagrange equations) or via the Ericksen–Leslie equations with the velocity assumed to be zero. However, here we choose to follow Laidlaw [13] and allow for solutions with non-zero velocity; in such cases it is essential that the full Ericksen–Leslie equations be employed. This approach has the additional advantage that one may be able to establish that an exchange of stabilities occurs; this would then preclude the possibility of an oscillatory instability, as found, for example, in some of the cases considered by Laidlaw [13].

2. The continuum equations

Here we summarize the equations proposed by Ericksen [7] and Leslie [8] to describe the isothermal behaviour of incompressible nematic liquid crystals. In Cartesian tensor notation the relevant equations for determining the velocity vector field **v** and the director **n** are the constraints

$$v_{i,i} = 0, \quad n_i n_i = 1, \tag{1}$$

together with the balance laws

$$\rho \dot{v}_i = -p_{,i} - \left(\frac{\partial W}{\partial n_{k,j}} n_{k,i}\right)_{,j} + \tilde{t}_{ij,j} + F_i, \tag{2}$$

$$0 = \gamma n_i + \left(\frac{\partial W}{\partial n_{i,j}}\right)_{,j} - \frac{\partial W}{\partial n_i} + \tilde{g}_i + G_i, \tag{3}$$

and the constitutive equations

$$\left.\begin{aligned} \tilde{t}_{ij} &= \alpha_1 n_k n_p A_{kp} n_i n_j + \alpha_2 n_j N_i + \alpha_3 n_i N_j + \alpha_4 A_{ij} \\ &\quad + \alpha_5 n_j n_k A_{ki} + \alpha_6 n_i n_k A_{kj}, \\ \tilde{g}_i &= -\gamma_1 N_i - \gamma_2 A_{ik} n_k, \end{aligned}\right\} \tag{4}$$

where

$$A_{ij} = \tfrac{1}{2}(v_{i,j} + v_{j,i}), \quad N_i = \dot{n}_i - \tfrac{1}{2}(v_{i,j} - v_{j,i})n_j, \tag{5}$$

a dot denoting the material derivative. Here ρ is the density; a micro-inertial term in (3) has been neglected, as is commonly done. The quantities p and γ are arbitrary scalar functions arising from the constraints (1), while **F** represents any external body force present and **G** any generalized external director body force acting. W is the stored energy per unit volume, and we adopt the Oseen–Frank form

$$\begin{aligned} 2W &= k_{22} n_{i,j} n_{i,j} + (k_{11} - k_{22} - k_{24}) n_{i,i} n_{j,j} \\ &\quad + k_{24} n_{i,j} n_{j,i} + (k_{33} - k_{22}) n_i n_j n_{k,i} n_{k,j}. \end{aligned} \tag{6}$$

In the present context the material parameters are constants satisfying the relations $\gamma_1 = \alpha_3 - \alpha_2$ and $\gamma_2 = \alpha_6 - \alpha_5$, and (from Ericksen [14]) the inequalities

$$\begin{aligned} k_{11} \geq 0, \quad k_{22} \geq 0, \quad k_{33} \geq 0, \quad |k_{24}| \leq k_{22}, \\ |k_{11} - k_{22} - k_{24}| \leq k_{11}, \end{aligned} \tag{7}$$

which in turn imply that

$$-k_{22} \leq k_{24} \leq \min\{k_{11}, k_{22}, 2k_{11} - k_{22}\} \tag{8}$$

(and that $k_{24} = k_{11}$ only if $k_{22} = k_{24} = k_{11}$). In the absence of information to the contrary, we must presumably allow for the possibility that $k_{24} < 0$.

Following Ericksen [15] we consider the resultant external body forces due to the presence of an applied magnetic field **H** to have the form

$$F_i = \{\chi_\perp H_p + \chi_a H_q n_q n_p\} H_{p,i}, \quad G_i = \chi_a H_p n_p H_i, \tag{9}$$

and the associated magnetic energy per unit volume W_f to be given by

$$2W_f = -\{\chi_\perp H_p H_p + \chi_a (H_p n_p)^2\}, \tag{10}$$

where $\chi_a = \chi_\parallel - \chi_\perp$, with $\chi_\parallel$ and $\chi_\perp$ denoting the constant magnetic susceptibilities parallel and perpendicular to the molecular axis, respectively. Henceforth we assume that

χ_a is strictly positive. Also, since the inclusion of gravity in calculations only leads to a modification in the pressure we ignore it in the following analysis.

To model weak anchoring at an interface S, one endows the interface with a surface energy w per unit area. Assuming that w depends only upon the director $\mathbf{n}$ and the outward surface normal $\boldsymbol{\nu}$, Jenkins and Barratt [10] derived the balance-of-couple condition

$$\frac{\partial W}{\partial n_{i,j}} \nu_j + \frac{\partial w}{\partial n_i} = \beta_1 n_i \quad \text{on} \quad S, \tag{11}$$

where β_1 is an arbitrary scalar and

$$w = w[(\boldsymbol{\nu} \cdot \mathbf{n})^2]. \tag{12}$$

(The Rapini-Papoular form [9] is $w = \frac{1}{2} w_0[1 - (\boldsymbol{\nu} \cdot \mathbf{n})^2]$, with w_0 a constant.) If in addition there is a preferred orientation $\mathbf{t}$ at the surface that minimizes w (sometimes called the 'easy axis') then we still employ (11) but with

$$w = w[(\boldsymbol{\nu} \cdot \mathbf{n})^2, (\mathbf{t} \cdot \mathbf{n})^2]. \tag{13}$$

3. Solution of the non-linear equilibrium equations

We examine the situation in which a sample of nematic liquid crystal confined between two coaxial circular cylinders of radii r_1 and r_2, where $r_2 > r_1$, is weakly anchored at the inner cylinder and strongly anchored at the outer cylinder. We assume that the initial director orientation is uniformly parallel to the axis of the cylinders, and consider the application of an azimuthal magnetic field $\mathbf{H}$. Choosing cylindrical polar coordinates r, ϕ, z such that the z axis and the cylinders' axis are coincident, we assume $\mathbf{H}$ depends only upon the radial coordinate r, so that the magnetic field has physical components

$$H_r = 0, \quad H_\phi = H/r, \quad H_z = 0, \tag{14}$$

where H is a constant. In the present situation it seems reasonable to seek director field solutions whose physical components have the form

$$n_r = 0, \quad n_\phi = \sin\theta(r), \quad n_z = \cos\theta(r), \quad 0 \le \theta \le \pi/2, \tag{15}$$

whereupon the problem reduces to solving the non-linear ordinary differential equation

$$k_{22}\frac{d^2\theta}{dr^2} + k_{22}\frac{1}{r}\frac{d\theta}{dr} + \{2(k_{22} - k_{33})\sin^2\theta - k_{22} + \chi_a H^2\}\frac{\sin\theta\cos\theta}{r^2} = 0, \tag{16}$$

subject to appropriate boundary conditions. Here we examine the situation in which there is strong anchoring on the outer cylinder, which requires

$$\theta(r_2) = 0, \tag{17}$$

and weak anchoring on the inner cylinder, with w having the form (12), whereupon the condition (11) reduces to

$$k_{22}\frac{d\theta}{dr} - k_{24}\frac{\sin\theta\cos\theta}{r} = 0 \quad \text{on} \quad r = r_1. \tag{18}$$

We note that with $\mathbf{n}$ restricted to lie in the local ϕ, z-plane, our choice for the functional form for the interfacial energy means that w takes the constant value $w(0)$ and so does not affect the balance-of-couple condition (11). We also make the observation that a solution of the form (15) with $-\pi/2 \le \theta \le 0$ parallels the one presented here.

It is immediately obvious that the uniform orientation pattern

$$\theta(r) = 0, \quad r_1 \le r \le r_2 \tag{19}$$

is a solution of (16) satisfying (17) and (18), and in the absence of any applied magnetic field also minimizes the total free energy of the system, ε, given by

$$\varepsilon = \int_V (W + W_f) dV + \int_{S_1} w dS, \tag{20}$$

where V is the volume occupied by the sample and S_1 is the bounding surface of the inner cylinder. However, distorted patterns of the form (15) are also possible and, with the change of variable

$$r = r_1 \exp(ls), \tag{21}$$

one seeks solutions of

$$\frac{d^2\theta}{ds^2} - 2\alpha\sin^3\theta\cos\theta + h\sin\theta\cos\theta = 0 \tag{22}$$

subject to the conditions

$$\theta = 0 \quad \text{on} \quad s = 1 \tag{23}$$

and

$$\frac{d\theta}{ds} = \beta\sin\theta\cos\theta \quad \text{on} \quad s = 0 \tag{24}$$

where

$$l = \ln\frac{r_2}{r_1}, \quad \alpha = l^2\left(\frac{k_{33}}{k_{22}} - 1\right), \quad \beta = l\frac{k_{24}}{k_{22}}, \quad h = l^2\left(\frac{\chi_a H^2}{k_{22}} - 1\right). \tag{25}$$

With (24), equation (22) integrates once to yield

$$\left(\frac{d\theta}{ds}\right)^2 = \beta^2\sin^2\theta_0\cos^2\theta_0 + \alpha(\sin^4\theta - \sin^4\theta_0) + h(\sin^2\theta_0 - \sin^2\theta) \equiv F(\theta, \theta_0, h), \tag{26}$$

where $\theta_0 = \theta(0)$. Since the right hand side of (26) must be non-negative for all possible values of θ, it follows that, for a continuous transition to occur, a necessary condition is

$$h + \beta^2 > 0, \quad \text{i.e.} \quad \chi_a H^2 > (k_{22}^2 - k_{24}^2)/k_{22}, \qquad (27)$$

which, with $(7)_4$, means that a distorted solution (15) is not possible in the absence of a magnetic field (in contrast with the situation considered by Palffy-Muhoray *et al.* [12]).

An inspection of (24) indicates that two distinct types of solution are possible, depending upon the sign of k_{24} and hence that of β. When

$$\beta < 0 \qquad (28)$$

it follows from (24) that the monotonic distortion

$$1 - s = \int_0^{\theta} \{F(\psi, \theta_0, h)\}^{-1/2} d\psi, \quad 0 \le s \le 1, \qquad (29)$$

is a solution of (22), (23) and (24), where h and θ_0 must satisfy the relation

$$1 = \int_0^{\theta_0} \{F(\theta, \theta_0, h)\}^{-1/2} d\theta. \qquad (30)$$

To determine the critical field H_c at which a smooth transition from the uniform axial orientation to a distorted orientation pattern is possible, one needs to consider the three cases

$$\text{(i) } \chi_a H_c^2 > k_{22}, \quad \text{(ii) } \chi_a H_c^2 = k_{22}, \quad \text{(iii) } 0 < \chi_a H_c^2 < k_{22} \qquad (31)$$

separately. Here, for the sake of brevity, we consider only case (i) in detail. Thus assuming h is strictly positive and employing the change of variable

$$\sin\theta = m \sin\lambda, \quad m = \left\{1 + \frac{\beta^2}{h}\cos^2\theta_0\right\}^{1/2} \sin\theta_0 \qquad (32)$$

in (30) we obtain the relationship

$$1 = \int_0^{\lambda_0} \{h\cos^2\lambda + \alpha m^2(\sin^4\lambda - \sin^4\lambda_0)\}^{-1/2} \times (1 - m^2\sin^2\lambda)^{-1/2}\cos\lambda \, d\lambda, \qquad (33)$$

where

$$\sin\lambda_0 = \frac{\sin\theta_0}{m} = \left\{1 + \frac{\beta^2}{h}\cos^2\theta_0\right\}^{-1/2}. \qquad (34)$$

Taking the limit $\theta_0 \to 0$ (so that $m \to 0$) one obtains the critical value H_c, at which a distorted solution is possible, to be given by

$$h^{1/2} = \sin^{-1}\left\{1 + \frac{\beta^2}{h}\right\}^{-1/2}. \qquad (35)$$

In an attempt to determine which of the two solutions (19) and (29) is the more likely to occur, it is common practice to compare the total energies associated with each in the expectation that the solution with least energy will be the one that is observed. Denoting by $\varepsilon(\theta_0)$ the energy associated with the distorted solution (29) and by $\varepsilon(0)$ the energy associated with the uniform axial alignment, one finds that, over a length L of the cylinders,

$$\begin{aligned}
\varepsilon(\theta_0) - \varepsilon(0) &= \frac{L}{2}\int_{r_1}^{r_2}\int_0^{2\pi}\left\{k_{22}\left(\frac{d\theta}{dr}\right)^2 - k_{24}\frac{\sin 2\theta}{r}\frac{d\theta}{dr} + k_{22}\frac{\sin^2\theta\cos^2\theta}{r^2} + k_{33}\frac{\sin^4\theta}{r^2} - \chi_a H^2\frac{\sin^2\theta}{r^2}\right\} r\,d\phi\,dr \\
&= \frac{\pi L k_{22}}{l}\int_0^{\theta_0}\{\beta^2\sin^2\theta_0\cos^2\theta_0 + h(\sin^2\theta_0 - 2\sin^2\theta) + \alpha(2\sin^4\theta - \sin^4\theta_0)\}\{F(\theta, \theta_0, h)\}^{-1/2} d\theta \\
&\quad + \tfrac{1}{2}\pi L k_{24}(1 - \cos 2\theta_0). \qquad (36)
\end{aligned}$$

As in the weak anchoring problems investigated by Barratt and Fraser [16], conditions ensuring that $\varepsilon(\theta_0) < \varepsilon(0)$ and that θ_0 increases monotonically with H, for fixed l, can be readily obtained only in the neighbourhood of $\theta_0 = 0$. Since the linear stability analysis of § 4 guarantees the onset of a continuous distortion as H exceeds H_c we do not persevere with such calculations here.

If on the other hand

$$\beta > 0 \qquad (37)$$

then a monotonic solution is not possible, since $d\theta/ds$ must vanish at least once in the layer. With the assumption that there is only one extremum in any distorted configuration, a possible solution is

$$1 - s = \begin{cases} \displaystyle\int_0^{\theta} \{F(\psi, \theta_0, h)\}^{-1/2} d\psi, & d\theta/ds \le 0, \\ \displaystyle\int_0^{\theta_m} \{F(\psi, \theta_0, h)\}^{-1/2} d\psi - \int_{\theta_m}^{\theta} \{F(\psi, \theta_0, h)\}^{-1/2} d\psi, & d\theta/ds \ge 0, \end{cases} \qquad (38)$$

where

$$h(\sin^2\theta_m - \sin^2\theta_0) = \beta^2\sin^2\theta_0\cos^2\theta_0 + \alpha(\sin^4\theta_m - \sin^4\theta_0) \qquad (39)$$

and θ_m is the value attained by θ at its extremum. Thus the critical value of h is given by the relationship

$$1 = \lim_{\theta_m, \theta_0 \to 0}\left\{\int_0^{\theta_m}\{F(\theta, \theta_0, h)\}^{-1/2} d\theta - \int_{\theta_m}^{\theta_0}\{F(\theta, \theta_0, h)\}^{-1/2} d\theta\right\}, \qquad (40)$$

which yields

$$h^{1/2} = \pi - \sin^{-1}\left\{1 + \frac{\beta^2}{h}\right\}^{-1/2}. \quad (41)$$

We recall that in the above analysis we have only considered the case when $\chi_a H^2$ exceeds k_{22}. Solutions for the other two cases in (31) can be obtained similarly. All these solutions could prove useful in the analysis of cylindrical samples of liquid crystal which have an isotropic core or cavity. However in the determination of elastic constants and anchoring coefficients it is the critical phenomena that are perhaps of more interest, and so we proceed, in the next section, to present a linear stability analysis of the problem.

It must be admitted that we have restricted the number of possible solutions by confining θ to lie between 0 and $\pi/2$ and by allowing non-monotonic distortions to have only one extremum. However the experience of Dafermos [17] suggests that relaxing these conditions only leads to distorted configurations that are associated with higher energies than those examined here.

4. A linear stability analysis

A uniform static axial orientation, with director and velocity of the form

$$\mathbf{n}_0 = (0, 0, 1), \quad \mathbf{v}_0 = \mathbf{0}, \quad (42)$$

is one obvious solution of the continuum equations (1)–(3) and the appropriate boundary conditions for the physical problem described in § 3. We now consider the stability of this basic state with respect to small amplitude perturbations $\mathbf{n}_1$ and $\mathbf{v}_1$ that have physical components of the form

$$\begin{aligned} \mathbf{n}_1 &= (n_r(r), n_\phi(r), 0)\exp(\sigma t), \\ \mathbf{v}_1 &= (0, v_\phi(r), v_z(r))\exp(\sigma t), \end{aligned} \quad (43)$$

and for which the no slip condition is satisfied on $r = r_1$ and $r = r_2$, a weak anchoring condition is satisfied on $r = r_1$ and a strong anchoring condition is satisfied on $r = r_2$. The representations (43) satisfy equations (1) identically, while a linearization of (2) and (3) results in a substantial uncoupling, and yields the equation for determining n_ϕ as

$$\frac{d^2 n_\phi}{dr^2} + \frac{1}{r}\frac{dn_\phi}{dr} + \left(\frac{h}{l^2 r^2} + \frac{\gamma_1 \sigma}{k_{22}}\right) n_\phi = 0. \quad (44)$$

A utilization of the change of variable (21) reduces the problem to that of solving the second-order linear differential equation

$$\frac{d^2 n_\phi}{ds^2} + (h + \delta\sigma e^{2ls}) n_\phi = 0, \quad (45)$$

subject to the boundary conditions

$$\frac{dn_\phi}{ds} = \beta n_\phi \quad \text{on} \quad s = 0, \quad n_\phi = 0 \quad \text{on} \quad s = 1, \quad (46)$$

where

$$\delta = l^2 r_1^2 \gamma_1 / k_{22}. \quad (47)$$

Multiplying (45) by $\bar{n}_\phi$, the complex conjugate of n_ϕ, and integrating the resulting equation from $s = 0$ to $s = 1$ yields

$$\int_0^1 \bar{n}_\phi \frac{d^2 n_\phi}{ds^2} ds + \int_0^1 (h + \delta\sigma e^{2ls})|n_\phi|^2 ds = 0. \quad (48)$$

An integration by parts of the first integral in (48) and use of (46) now gives the relation

$$-\beta |n_\phi|^2\Big|_{s=0} - \int_0^1 \left\{\left|\frac{dn_\phi}{ds}\right|^2 - (h + \delta\sigma e^{2ls})|n_\phi|^2\right\} ds = 0, \quad (49)$$

from which we conclude that σ must be real. It therefore follows that the principle of exchange of stabilities holds, which means that critical values for the onset of instability are given when σ is identically zero. Thus, for determining critical phenomena, we are only interested in solutions of

$$\frac{d^2 n_\phi}{ds^2} + h n_\phi = 0, \quad (50)$$

subject to the boundary conditions (46). This is a classical type of eigenvalue problem and for a non-trivial solution one finds that

$$-\beta = \begin{cases} (-h)^{1/2}\coth((-h)^{1/2}), & h < 0 \\ 1, & h = 0 \\ h^{1/2}\cot(h^{1/2}), & h > 0 \end{cases} \quad (51)$$

with $\beta, h(\geq -l^2)$ and l as given in (25). For a fixed value of l, equation (51) gives the relationship between k_{24} and the critical magnetic field H_c at which one anticipates the onset of a Freedericksz transition. Thus the measurement of H_c will, assuming that k_{22} and χ_a are known material parameters, determine the material parameter k_{24}. The relationship (51) is illustrated pictorially in figure 1 (and we remark that $h \to -\infty$ as $\beta \to -\infty$, that $h \to \pi^2$ as $\beta \to +\infty$, and that $h = \pi^2/4 \approx 2{\cdot}47$ when $\beta = 0$). This curve is multi-branched; however, since β is restricted by $(7)_4$, it follows that the only portion of the curve that is relevant (for a given l) lies in

$$-l \leq \beta \leq l, \quad h_1 \leq h \leq h_2, \quad (52)$$

where h_1 and h_2 are defined by

$$\left.\begin{aligned} l &= -h_2^{1/2}\cot(h_2^{1/2}), && \tfrac{1}{4}\pi^2 < h_2 < \pi^2 && \forall l > 0, \\ l &= h_1^{1/2}\cot(h_1^{1/2}), && 0 \leq h_1 < \tfrac{1}{4}\pi^2 && \text{if } l \leq 1, \\ l &= (-h_1)^{1/2}\coth((-h_1)^{1/2}), && h_1 \leq 0 && \text{if } l \geq 1. \end{aligned}\right\} \quad (53)$$

On this section of the curve, k_{24} is a continuous, single-valued, monotonic increasing function of H_c. One notes that the curve of β versus h is the same whatever

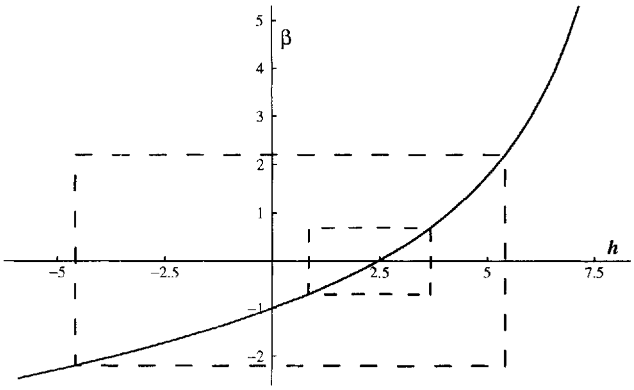

Figure 1. Sketch of β ($:= lk_{24}/k_{22}$) as a function of the (reduced) critical magnetic field h ($:= l^2(\chi_a H_c^2/k_{22} - 1)$), as given by equation (51), where $l := \ln(r_2/r_1)$. The rectangles illustrate the regions of the h, β-plane that are relevant for typical values of l satisfying $l < 1$ (smaller rectangle) and $l > 1$ (larger rectangle).

the value of l: changing l merely changes the portion (52) of this curve that is relevant.

From the diagram one can see that for $l < 1$ the critical value of h must be positive. The non-linear solution considered in detail in § 3 corresponds to the case $h > 0$, and we observe that equation (35) relating k_{24} and H_c is equivalent to (51) in this case.

In the alternative situation where there is weak anchoring at the outer cylinder and strong anchoring at the inner cylinder, a similar analysis again yields the critical relationship (51) but with left hand side replaced by $+\beta$. When strong anchoring obtains at both boundaries, one finds the critical magnetic field to be given by

$$h = \pi^2, \quad \text{i.e.} \quad \chi_a H_c^2/k_{22} = 1 + (\pi/l)^2, \qquad (54)$$

which agrees with the result obtained by Atkin and Barratt [6]. With weak anchoring at both boundaries the critical field is given by

$$h = -\beta^2, \quad \text{i.e.} \quad \chi_a H_c^2/k_{22} = 1 - (k_{24}/k_{22})^2, \qquad (55)$$

which, we observe, is independent of l.

Lastly we consider briefly the consequence of modelling weak anchoring via the more general expression (13) for the surface energy, rather than (12). If, for simplicity, the easy axis $\mathbf{t}$ is taken to be the unit axial vector, then the boundary condition (46), is replaced by

$$\frac{dn_\phi}{ds} = (\beta - U_1)n_\phi \quad \text{on} \quad s = 0, \qquad (56)$$

where

$$U_1 = \frac{2lr_1}{k_{22}} \frac{\partial w}{\partial t_n}\bigg|_{t_n = 1}, \qquad t_n = (\mathbf{t}\cdot\mathbf{n})^2. \qquad (57)$$

With strong anchoring at the boundary $s = 1$, one clearly again obtains the critical conditions (51), but with β replaced by $\beta - U_1$, i.e. the above results again hold, but with the quantity lk_{24}/k_{22} modified by the (dimensionless) anchoring coefficient U_1. A plot of $\beta - U_1$ versus h is as in figure 1, and, with the assumption that U_1 is negative, the relevant portion of the curve is again given simply by (52), but with l replaced by $l + |U_1|$ in $(53)_1$ and by $l - |U_1|$ in $(53)_{2,3}$.

Similarly for weak anchoring on $r = r_2$ the appropriate condition is

$$\frac{dn_\phi}{ds} = (\beta + U_2)n_\phi \quad \text{on} \quad s = 1, \quad U_2 = \frac{2lr_2}{k_{22}} \frac{\partial w}{\partial t_n}\bigg|_{t_n = 1}, \qquad (58)$$

and again (51) holds, but with $-\beta$ on the left hand side replaced by $+(\beta + U_2)$.

5. Discussion and concluding remarks

We have examined what appears to be a feasible experimental arrangement in which a uniform axially aligned nematic liquid crystal confined between concentric circular cylinders is subjected to an applied azimuthal magnetic field. The analysis predicts that the axial state will persist only until a critical field strength H_c is exceeded. With a weak anchoring condition of the form (24) on either cylindrical boundary, one finds that H_c depends on the surface elastic constant k_{24} via relations like (35), (41) or (51). Thus observations of critical phenomena could, in principle, provide a means of determining k_{24}. For a more general weak anchoring condition of the form (56), measurements of H_c from two different set-ups, with two different values of r_1, should yield information for determining both k_{24} and the anchoring coefficient U_1.

We can estimate the critical axial current J_c in the annulus that produces the threshold field $H_\phi = H_c/r$, as follows [18]. In the absence of time-varying electric fields, it follows from Maxwell's equations that $J_c = 2\pi H_c$. But, in the case of weak anchoring at both boundaries, equation (55) gives the order-of-magnitude estimate $H_c \sim (k_{22}/\chi_a)^{1/2}$, so that $J_c \sim 2\pi(k_{22}/\chi_a)^{1/2}$. With typical values for the material constants, one finds then that $J_c \sim 20$ A. (In addition, the results of Jenkins and Barratt [10] suggest that if the interfacial energy strongly prefers an azimuthal orientation at the boundary, then the critical field strength, and the associated current, may be significantly reduced.)

The critical relationships derived from the non-linear static theory and the linear stability analysis based on the dynamical equations are in agreement. The linear theory provides a proof (via the 'exchange of stabilities' argument) that an instability must occur for $H > H_c$; to deduce this within the non-linear static theory, one must apparently make additional assumptions (for example, that any other static distorted state will have a higher energy associated with it). In addition, the linear theory shows that the component n_r of $\mathbf{n}_1$ in (43) is of smaller order than n_ϕ, which implies that the instability that arises for $H \geq H_c$ will involve no twisting of the director out of the local ϕ, z-plane, at least initially; this lends support to the choice of the form of solution (15).

Of course, only the non-linear theory is capable of describing how any distortion will eventually develop when $H \geq H_c$ (though the shape of the eigenfunctions from the linear theory may also give some indication of this). It is therefore interesting that the non-linear theory predicts that at least two distinct types of solution are possible, and that for an interfacial energy of the form (12), the type of solution that occurs depends on the sign of k_{24}. Further solutions of the form (29)–(30) or (38)–(40) may prove useful in interpreting the behaviour of a nematic in the vicinity of polymer fibres, or of a nematic contained in the annular gap between a rigid cylinder and an isotropic core.

We could equally have considered the situation where the applied field is radial, of the form $\mathbf{H} = (H/r, 0, 0)$, with H a constant; it then transpires that the critical relation (51) again holds, but with h and β now defined by

$$h = l^2\left(\frac{\chi_a H^2}{k_{11}} - 1\right) \quad \text{and} \quad \beta = \frac{l}{k_{11}}(k_{24} + k_{22} - k_{11} + U_3) \tag{59}$$

respectively, where

$$U_3 = \frac{2lr_1}{k_{11}} \frac{\partial w}{\partial v_n}\bigg|_{v_n = 0}, \quad v_n = (\mathbf{v}\cdot\mathbf{n})^2. \tag{60}$$

Although such a magnetic field may be more difficult to establish experimentally, the analogous set-up involving a radially applied electric field should be easier to achieve, and the critical electric field strength satisfies relations precisely analogous to those for the magnetic field.

Alternative feasible arrangements might involve a sample aligned azimuthally with a field applied axially, or a sample aligned radially with a field applied axially or azimuthally. However, additional mathematical complications arise for these set-ups; some of these complications, evident in the paper by Palffy-Muhoray *et al.* [12], are associated with the fact that the initial equilibrium state is non-uniform, and purely mechanical instabilities can arise (which again permit a reduction in the field strength needed to induce a transition). Other complications arise in the linear stability analysis. In particular, in the case of an azimuthal configuration with the applied field axial, the equation that corresponds to (50) here has non-constant coefficients, and the critical field is not so straightforwardly obtainable. Analysis of this and similar set-ups are currently being pursued.

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References

[1] ONDRIS-CRAWFORD, R. J., CRAWFORD, G. P., ŽUMER, S., and DOANE, J. W., 1993, *Phys. Rev. Lett.*, **70**, 194.
[2] POLAK, R. D., CRAWFORD, G. P., KOSTIVAL, B. C., DOANE, J. W., and ŽUMER, S., 1994, *Phys. Rev.*, **49**, R978.
[3] SPARAVIGNA, A., LAVRENTOVICH, O. D., and STRIGAZZI, A., 1994, *Phys. Rev. E*, **49**, 1344.
[4] KRALJ, S., and ŽUMER, S., 1993, *Liq. Crystals*, **15**, 521.
[5] LESLIE, F. M., 1970, *J. Phys. D: Appl. Phys.*, **3**, 889.
[6] ATKIN, R. J., and BARRATT, P. J., 1973, *QJMAM*, **26**, 109.
[7] ERICKSEN, J. L., 1961, *Trans. Soc. Rheol.*, **5**, 23.
[8] LESLIE, F. M., 1968, *Arch. Rat. Mech. Anal.*, **28**, 265.
[9] RAPINI, A., and PAPOULAR, M., 1969, *J. de Physique, Colloq.*, **30**, C4–54.
[10] JENKINS, J. T., and BARRATT, P. J., 1974, *QJMAM*, **27**, 111.
[11] BARRATT, P. J., 1974, *QJMAM*, **27**, 505.
[12] PALFFY-MUHORAY, P., SPARAVIGNA, A., and STRIGAZZI, A., 1993, *Liq. Crystals*, **14**, 1143.
[13] LAIDLAW, W. G., 1979, *Phys. Rev. A*, **20**, 2188.
[14] ERICKSEN, J. L., 1966, *Phys. Fluids*, **9**, 1205.
[15] ERICKSEN, J. L., 1962, *Arch. Rat. Mech. Anal.*, **9**, 371.
[16] BARRATT, P. J., and FRASER, C., 1983, *Proc. Edinburgh Math. Soc.*, **26**, 319.
[17] DAFERMOS, C. M., 1968, *SIAM J. Appl. Math.*, **16**, 1305.
[18] We thank an anonymous referee for pointing this out to us.